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A New Approach to the Head-Tail Instability

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ABSTRACT

In this paper we will give a new approach of the head-tail theory and apply it to coherent transverse instability induced by matched kicker magnet. The theoretical results agree with the Pellegrini-Sands formula of the instability in the limit of small chromaticity, and also explain completely the experimental results on the instability observed in KEK booster.

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#1. INTRODUCTION

The head-tail theory developed by Pellegrini and Sands on coherent transverse instability of bunched beam explains quite well the experimental features of the instability especially with respect to the chromaticity dependence.[1,2] But numerical agreement has not yet been obtained because of the uncertainty of wake field. Recently we have found that ferrite loaded kicker magnet induces the instability in KEK booster,[3] and that free string model is convenient to describe the wake field induced by the beam.[4] In this paper we will develop a theory with a new approach to the head-tail instability with the use of the free string model in regard to the instability induced by a matched kicker magnet.

In the head-tail theory by Pellegrini and Sands, the circulation of particles within the bunch plays the essential role. The head particle induces wake field, which increases the betatron oscillation amplitude of the tail particle, then the head and tail particle exchange the position. But the free string model does not include this circulation mechanism but for the amplitude modulation by the synchrotron motion. Our idea is that when the circulation is fast enough compared with the build up of the instability, the particles kicked by the wake field redistribute in a normal form, which is given later by (2-1), within one revolution period. Thus the wake field is always induced by the normal form with the betatron oscillation amplitude increasing every revolution.

In the next section we give the new formalism of the growth rate of the instability. Application is made in section 3 for the instability induced by matched kicker magnet. Transient process of the induced current in the magnet is explained. The Pellegrini-Sands formula on the head-tail effect is derived as a limit of small chromaticity. It is shown in section 4 that the numerical results explain completely the experimental results on the instability observed in KEK booster. Details of this paper will be published later.[5]

#2. FORMALISM OF GROWTH RATE

According to the free string model of the bunched beam,[4] the betatron oscillation of the l-th part of the string is expressed by

$$x_\ell = Z_\ell \, e^{i(\omega_\beta kT + \omega_\xi \tau_\ell)} \quad (k\text{=integer}) \tag{2-1}$$

where Z_l is the amplitude, ω_β the betatron oscillation frequency, T the revolution period, ω_ξ the chromatic ferquency and τ_ℓ the time of arrival at a given point of the ring. In the model every part of the string rotates with the same frequency and does not make the synchrotron oscillation. But this oscillation is taken into account in (2-1) by the phase modulation $e^{i\omega_\xi \tau_\ell}$. Dynamic motion of the free string given by (2-1) is illustrated for the case of $\omega_\xi \tau = 2\pi$, and 0 in Fig. 1, where τ is the bunch length in time. The figure presents the string motion to be observed at a given point in the ring. As seen in the figure, we should take τ_ℓ as time variable.

Let's consider the betatron motion of the s-th part of the string. We can expand Z_s in a Fourier series with respect to τ_s. For simplicity we consider only one term of the series;

$$Z_s = A_m \cos m\omega_1 \tau_s \quad (m\text{:integer}) \tag{2-2}$$

where A_m is amplitude, m the oscillation mode and $\omega_1 = 2\pi/\tau$. Next define S_1 in the phase space

$$S_1{}^2 = (\mathrm{Re}(x_s))^2 + (\mathrm{Re}\,(\dot{x}_s/\omega_\beta))^2 \tag{2-3}$$

where $\mathrm{Re}(x_s)$ means the real part of x_s and $\dot{x}_s = \frac{dx_s}{dt'}$, where $t' \approx kT$. From (2-2) and (2-3) we obtain

$$\sqrt{\langle S_1^2 \rangle_{\tau_s}} = \begin{cases} Am/\sqrt{2} & \text{for } m \neq 0 \\ Ao & \text{for } m = 0 \end{cases} \tag{2-4}$$

where $\langle\ \rangle_{\tau_s}$ means the average over $-\tau/2 \leq \tau_s \leq \tau/2$. When the beam is kicked by the amount $\Delta\dot{x}_s$ at a place of the ring, we get

$$S_2^{\,2} = (\mathrm{Re}[x_s]^2 + (\mathrm{Re}[(\dot{x}_s + \Delta\dot{x}_s)/\omega_\beta])^2 \tag{2-5}$$

therefore we get

$$\sqrt{\langle S_1^{\,2}\rangle_{\tau_s}} = \frac{Am}{\sqrt{2}} - \frac{2}{\omega_\beta} < \cos\,(m\omega_1\,\tau_s)\ \sin\,(\omega_\beta t' + \omega_\xi\tau_s)\ X$$

$$\mathrm{Re}[\Delta\dot{x}_s] >_{\tau_s} \tag{2-6}$$

The difference between (2-6) and (2-4) gives the increase of the amplitude,

$$\Delta\, A_m = \frac{-2}{\omega_\beta} < \cos\,(m\omega_1\tau_s)\ \sin\,(\omega_\beta t' + \omega_\xi\tau_s)\ \mathrm{Re}[\Delta\dot{x}_s] >_{\tau_s} \tag{2-7}$$

where factor 2 should be replaced by 1 for mode m = 0. The $\Delta\, A_m$ is the increase per one kick or per one revolution. If Re $[\Delta\dot{x}_s]$ is proportional to A_m we get an exponential increase of the amplitude because we have

$$\frac{d\, A_m}{d\,t'} \approx \frac{\Delta\, A_m}{T} = \beta_m\, A_m \tag{2-8}$$

the growth rate is given by

$$\beta_m = -\,\frac{2}{\omega_\beta T} < \cos\,(m\omega_1\tau_s)\ \sin\,(\omega_\beta t' + \omega_\xi\tau_s)\mathrm{Re}[\Delta\dot{x}] >_{\tau_s}$$

$$\mathrm{Re}[\Delta\dot{x}_s) > \tag{2-9}$$

The build-up time is defined by $\tau_\gamma = 1/\beta_m$.

#3. APPLICATION TO THE INSTABILITY INDUCED BY MATCHED KICKER MAGNET

3-1 Transient process of induced current.

In KEK Booster the instability is induced by the interaction of the beam with the kicker magnet for fast beam extraction. The magnet is composed of the lamination of ferrite cores and electric plates, and connected with matched cable and matched resistance. There are some losses in ferrite cores. Hence the magnet can be expressed with an infinite series of LCR ladder network. When a beam current passes through the gap of the magnet, secondary current is induced in the one turn coil of the magnet like transformer. Because of the C-type structure of the magnet, the mutual inductance depends on the horizontal position x(mm) of the passing current;

$$M = a - bx \tag{3-1}$$

where $a = 1.01 \times 10^{-7}$ and $b = 1.60 \times 10^{-9}$. Therefore total equivalent circuit is given as shown in Fig. 2.

Now suppose that a step-up current passes through the gap. Then a series of ring currents is induced in every mesh of the magnet instantaneously and simultaneously as shown in Fig. 2. In the next instance it begins to travel in both directions with a constant velocity. The traveling time through the magnet is given by

$$\tau_0 = N\sqrt{LC} \doteqdot 42 \text{ nsec} \tag{3-2}$$

where N is the number of meshes. The ring currents still remaining within the magnet decrease linearly;

$$\underline{\psi}\ (\Delta t) = \begin{cases} 1-\Delta t/\tau_0 & \text{for } 0 \leq \Delta t \leq \tau_0 \\ 0 & \text{otherwise} \end{cases} \tag{3-3}$$

3-2 induced current due to bunched beam

Assume following current distribution for the bunched beam,

$$i_\ell = i_0 \cos\left(\frac{\omega_1}{2}\tau_\ell\right),\ \left(\omega_1 = \frac{2\pi}{\tau}\right) \tag{3-4}$$

The ring current induced at the n-th mesh of the equivalent circuit should follow the equation,

$$\delta \frac{\partial I_n}{\partial \tau_\ell} + \frac{\partial^2 I_n}{\partial \tau_\ell{}^2} = \frac{1}{L}\frac{\partial^2}{\partial \tau_\ell{}^2}(Mi_\ell) \tag{3-5}$$

where $\delta = R/L$. Inserting (2-1), (2-2), (3-1) and (3-4) into (3-5) we get the induced current due to the l-th part

$$\begin{aligned} I_\ell = & \sum_{\pm} \frac{ai_0}{2L} D_m^{\pm} e^{j\Theta_m^{\pm}} e^{\pm j\frac{\omega_1}{2}\tau_\ell} \\ & - \sum_{\pm} A_m e^{j\omega_\beta t'} \frac{bi_0}{4L} D_m^{\pm} e^{j\Theta_m^{\pm}} e^{j\Omega_m^{\pm}\tau_\ell} \end{aligned} \tag{3-6}$$

where

$$D_m^{\pm} = 1/\sqrt{1 + (\delta/\Omega_m^{\pm})^2}$$

$$\Theta_m^{\pm} = \arctan\,(\delta/\Omega_m^{\pm}) \tag{3-7}$$

$$\Omega_m^{\pm} = \begin{cases} \omega_\xi \pm (m \pm 1/2)\,\omega_1 & \text{for b-term} \\ \pm\,\omega_{1/2} & \text{for a-term} \end{cases}$$

It is easily shown that only the second term of (3-6) contributes to the instability. The δ depends much on frequency. This was obtained experimentally and $D(\omega)$ and $\Theta(\omega)$ are shown in Fig. 3.

3-3 Growth Rate of the Instability

The series of the ring current in the one turn coil produces magnetic field vertically. The average field in the magnet, which is induced by the l-th part, is given by

$$B_\ell\,(\Delta t) = \frac{\mu_o}{h}\, I_\ell\, \bar{\Psi}\,(\Delta t) \tag{3-8}$$

where μ_o is the vacuum permeability, h the gap height of the magnet and Δt the time experience. This field kicks the following s-th part by the amount

$$\Delta\dot{x}_s = -\frac{e\ell_B}{m_o\gamma}\,\frac{\mu_o}{h}\,\frac{bi_o}{4L}\,Am\,e^{j\omega_\beta t'}\,\times$$

$$\sum_{\pm} D_m^{\pm}\,e^{j\Theta_m^{\pm}}\sum_{\ell}^{\ell<s} e^{j\Omega_m^{\pm}\tau_\ell}\,\bar{\Psi}\,(\tau_{s\ell}) \tag{3-9}$$

where l_B is the magnet length, m_o the particle mass, γ the relativistic energy and $\tau_{s\ell} = \tau_s - \tau_\ell$. The summation $\sum_{\ell}^{\ell<s}$ means the contribution of all the parts which precede the s-th part. The summation is obtained by the following integration,

$$\sum_{\ell}^{\ell<s} = \frac{1}{\tau} \int_{-\tau/2}^{\tau_s} e^{j\Omega \overset{\pm}{m} \tau_\ell} \; \underline{\Psi} \; (\tau_{s\ell}) \, d\tau_\ell$$

$$= \left[\frac{1}{j\Omega \overset{\pm}{m} \tau} \left\{ 1 - \left(1 - \frac{t''}{\tau_o}\right) e^{-j\Omega \overset{\pm}{m} t''} \right\} \right.$$

$$\left. + \frac{1}{\Omega \overset{\pm 2}{m} \tau_o \tau} \left(1 - e^{-j\Omega \overset{\pm}{m} t''} \right) \right] e^{j\Omega \overset{\pm}{m} \tau_s} \tag{3-10}$$

$$\equiv W (\Omega \overset{\pm}{m}) \; e^{j\Omega \overset{\pm}{m} \tau_s}$$

where

$$t'' = \begin{cases} \tau_s + \tau/2 & \text{for } -\tau/2 \leq \tau_s \leq \tau_o - \tau/2 \\ \tau_o & \text{for } \tau_o - \tau/2 \leq \tau_s \leq \tau/2 \end{cases}$$

Consequently the real part of $\Delta \dot{x}_s$ is given by

$$\mathrm{Re}\,(\Delta \dot{x}_s) \propto \left\{ \mathrm{Re}\,(W) \cos\,(\omega_\beta t' + \Omega \overset{\pm}{m} \tau_s + \Theta \overset{\pm}{m}) \right. \tag{3-11}$$

$$\left. - \mathrm{Im}\,(W) \sin\,(\omega_\beta t' + \Omega \overset{\pm}{m} \tau_s + \Theta \overset{\pm}{m}) \right.$$

where Im (W) is the imaginary part of W. Multiplying $\sin\,(\omega_\beta t' + \omega_\xi \tau_s)$ and taking the time average for t', we get finally the growth rate

$$\beta_m = -H^* \sum_{\pm} D_m^{\pm} \left\langle \cos\,(m\omega_1 \tau_s) \left\{ \mathrm{Re}\,(W) \sin\,[\pm (m \pm 1/2)\, \omega_1 \tau_s + \Theta_m^{\pm}] \right. \right. \tag{3-12}$$

$$\left. \left. + \mathrm{Im}\,(W) \cos\,[\pm (m \pm 1/2)\, \omega_1 \tau_s + \Theta_m^{\pm}] \right\} \right\rangle_{\tau_s}$$

where

$$H^* = \frac{1}{16} \; \frac{e^2 \mu_o}{m_o} \; \frac{b \ell_B}{hL} \; \frac{N_o}{\nu \gamma \tau} \tag{3-13}$$

where N_0 is the number of particles in the bunch. H* should be divided by 2 for mode m = 0. We used the relation $i_0 = \pi e N_0/2\tau$ for the current distribution (3-4). For sin-mode we can get a similar expression.

3-4 Derivation of the Pellegrini-Sands Formula

In a simplified case of the previous discussion, we can obtain the Pellegrini-Sands formula of the head-tail instability as the limit of small chromaticity. Assume R = 0, then $D_m^{\pm} = 1$ and $\Theta_m^{\pm} = 0$. Hence we get from (2-9) and (3-9)

$$\beta_m = H^* < \sum_{\pm} \sum_{\ell}^{\ell<s} \cos(m\omega_1\tau_s)\sin(\omega_\beta t' + \omega_\xi\tau_s) \times \cos(\omega_\beta t' + \Omega_m^{\pm}\tau_\ell)\,\Psi(\tau_{s\ell}) >_{\tau_s} \tag{3-14}$$

The summation $\sum_{\pm}$ and many turn average for t' lead to

$$\beta_m = 2H^* < \sum_{\ell}^{\ell<s} \cos(m\omega_1\tau_s)\cos(m\omega_1\tau_\ell)\sin\omega_\xi(\tau_s - \tau_\ell) \times \cos\left(\frac{\omega_1}{2}\tau_\ell\right)\Psi(\tau_{s\ell}) >_{\tau_s} \tag{3-15}$$

This expression gives clear meaning of each term. Now assume, further, constant current distribution ($\cos\frac{\omega_1}{2}\tau_\ell \longrightarrow 1$) and slow decay ($\Psi(\tau_{s\ell}) \longrightarrow 1$) but no effect on the next turn. Then we get by the summation $\sum_{\ell}^{\ell<s}$ and averaging over τ_s,

$$\beta_m = 2H^* \frac{\omega_\xi/\tau}{(m\omega_1)^2 - \omega_\xi^2}\left\{\left(\frac{1}{2}\right)^* + \frac{\omega_\xi^2}{(m\omega_1)^2 - \omega_\xi^2}\,\frac{\sin(\omega_\xi\tau)}{\omega_\xi\tau}\right\} \tag{3-16}$$

where (1/2)* should be replaced by 1 for m = 0. In the limit of $\omega_\xi\tau \ll 1$ we get

$$\beta_m = \begin{cases} -\dfrac{H^*}{3}\,\omega_\xi\tau & \text{for } m = 0 \\[2ex] \dfrac{H^*}{(2\pi m)^2}\,\omega_\xi\tau & \text{for } m \neq 0 \end{cases} \tag{3-17}$$

where $\omega_\xi = \frac{\xi}{\eta}\omega_\beta$, ξ is the chromaticity, $\eta = \alpha - \frac{1}{\gamma^2}$ and α the momentum compaction factor. This is very close to the Pellegrini-Sands formula;

$$\beta_m = \frac{2}{\pi^2} \frac{N_0 SA}{\alpha} \frac{\xi}{4m^2-1} \tag{3-18}$$

where $A = \tau/2$ and S is wake force divided by $m_0\gamma$.

#4. NUMERICAL CALCULATION AND COMPARISON WITH EXPERIMENT

We calculate numerically the growth rate (3-12) of the instability and show that the results explain completely the experimental results on the instability, which are summerized as follows:[3]

1. Build up time is $3 \sim 5$ msec.
2. Instability is induced at $13 \sim 20$ msec after beam injection.
3. Threshold beam intensity is about $N_0 = 3 \times 10^{11}$ ppp.
4. Only mode $m = 0$ has been observed.
5. In spite of the instability the emittance of the extracted beam does not seem increased.
6. Instability is suppressed by reducing the chromaticity to zero and also by increasing it about twice.
7. Instability is suppressed by applying a small amount of octupole correction field.

We note that ξ is positive at the first stage of acceleration, becomes zero at 20 msec after injection and goes to negative owing to magnet saturation, and that η is negative. Hence the chromatic frequency ω_ξ changes the sign from negative to positive at 20 msec.

Numerical results of the growth rate for the cos- and sin-mode are shown in Fig. 4, which explain items 1, 2, 4 and 5. The growth rate for mode $m = 0$ becomes maximum around 17 msec and rapidly decreasing, becomes negatively large. Therefore coherently increased amplitude rapidly damps coherently just after the instability, resulting in no emittance growth. Figure 4 is drawn for $N_0 = 4 \times 10^{11}$, thus the threshold intensity $N_0 = 3 \times 10^{11}$ gives critical growth rate $\beta_c = 135\ \text{sec}^{-1}$ shown with the dotted line in the figure. Figure 5 shows $q(= \frac{\xi}{\eta}\nu)$ and χ $(= \omega_\xi \tau)$ dependence of the growth rate at 16 msec after injection. The operating point is now q = -1.5 at 16 msec, which is corresponding to maximum β_0. Therefore increase or decrease of the chromaticity by the amount of $\Delta\xi \approx \pm \xi$ (at 16 msec) drives β_0 under the critical growth rate, being in accord with the item 6.

Tune spread smears out the coherent motion of the beam . The instability would be suppressed by the presence of the tune spread;

$$\Delta\nu \gtrsim T/\tau_\gamma \approx 3 \times 10^{-5} \tag{4-1}$$

In fact it was damped with correction octupole field $Bl_0 \approx \pm 20T/m^2$, which brings about the tune spread $\Delta\nu \approx 2 \times 10^{-5}$. The threshold or critical build-up time gives build-in tune spread about 2.7×10^{-5}, being extremely narrow. Momentum dependent tune spread is approximately 2×10^{-3} because the chromaticity is about 0.3. However we should notice that this tune spread does not smear out the coherent motion, but promotes the instability, because momentum dependent tune shift is related coherently with the modulation of the betatron oscillation amplitude as given by (2-1).

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FIGURE CAPTIONS

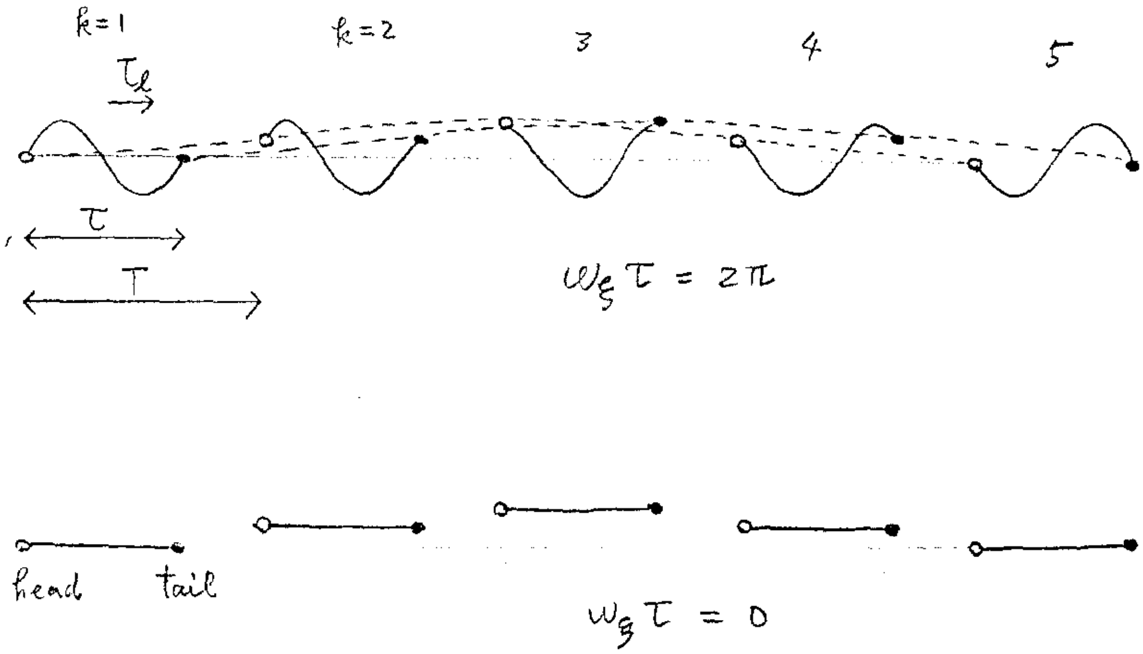
Fig. 1 String Motion given by (2-1).

Fig. 2 Equivalent circuit of matched kicker magnet. Resistence R, which expresses the ferrite loss, is not drawn.

Fig. 3 Frequency dependence of $D(\omega)$ and $\Theta(\omega)$.

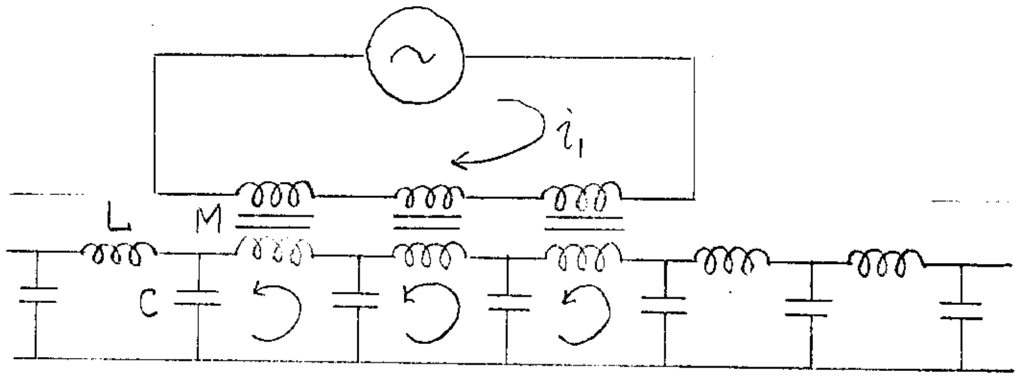
Fig. 4 Calculated growth rate for cos- and sin-mode during the acceleration.

Fig. 5 Chromaticity dependence of the growth rate for cos-and sin-mode.



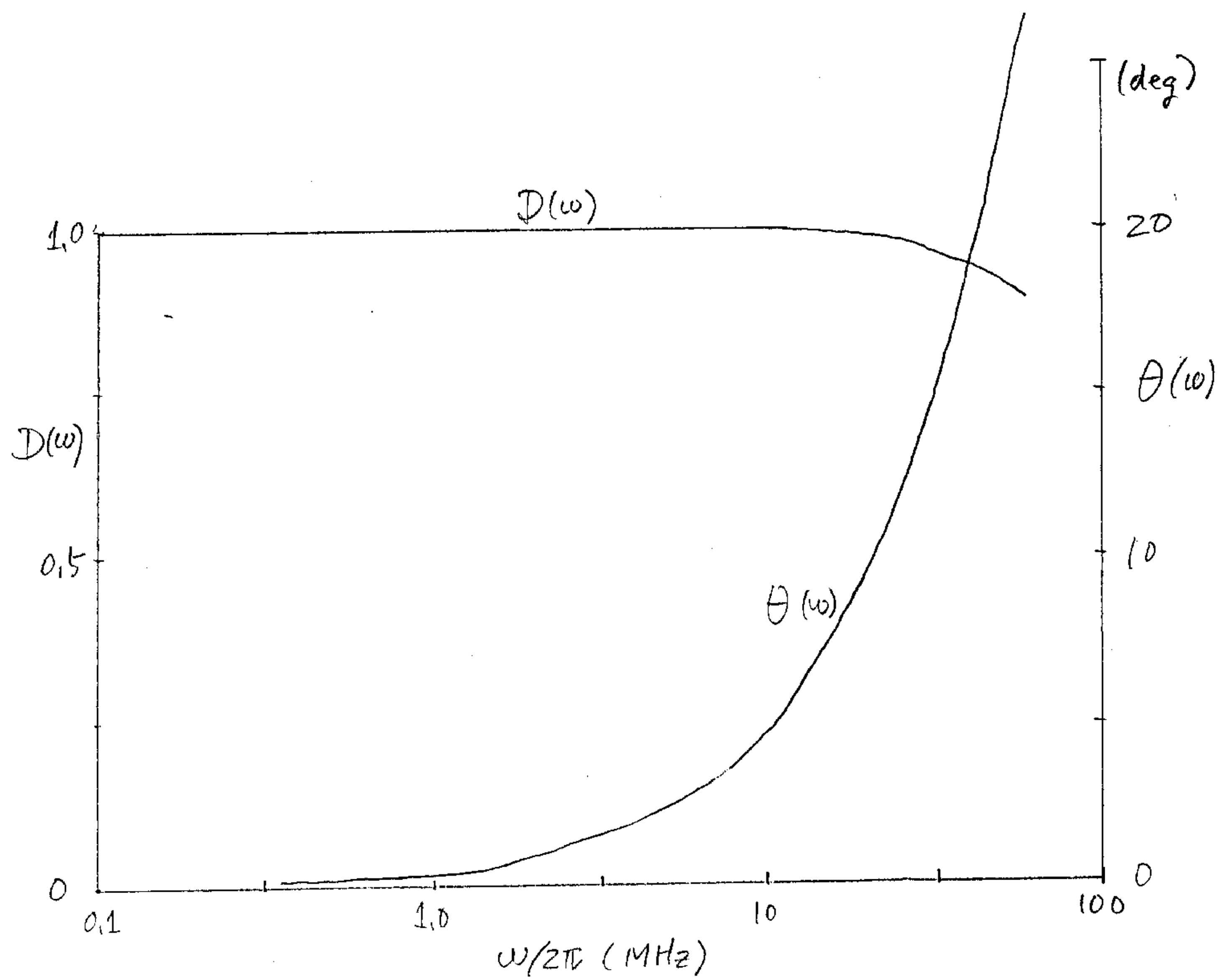
String Motion

Fig. 1.



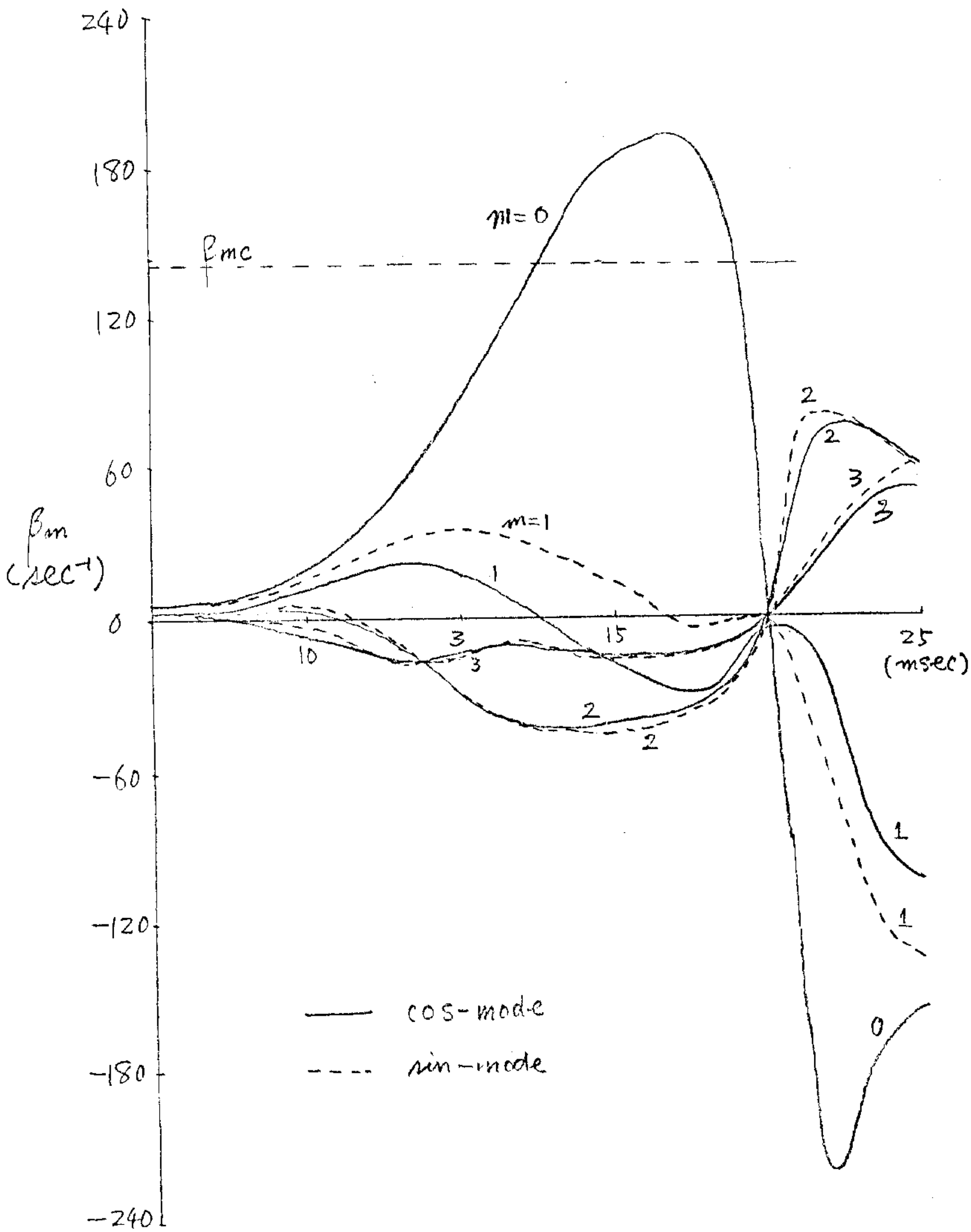
Equivalent Circuit of Matched Kicker Magnet.

Fig. 2.



Frequency Dependence of D(ω) and Θ(ω).

Fig. 3.



Calculated Growth Rate for Cos-and sin-mode During the Acceleration.

Fig. 4.

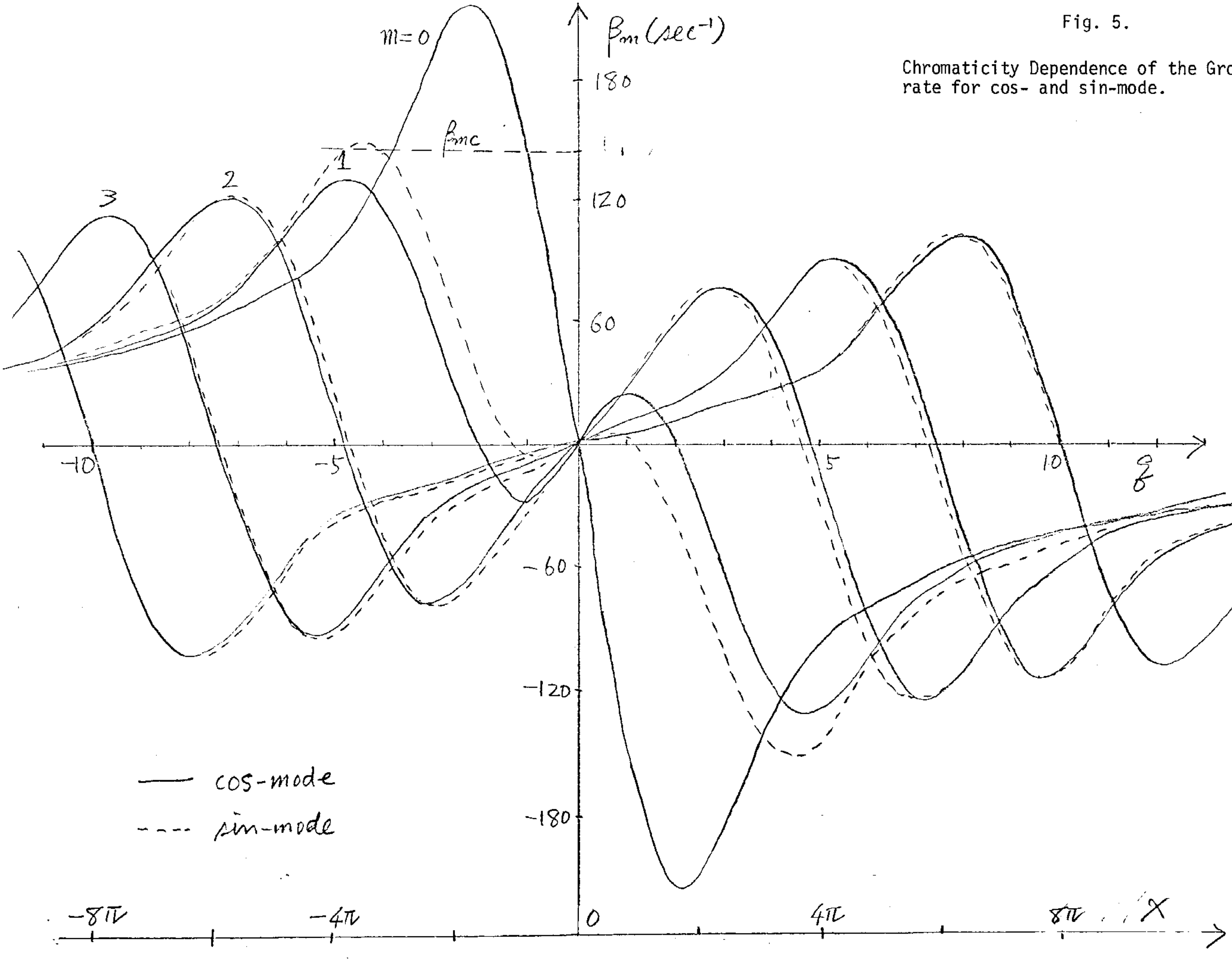


Fig. 5.

Chromaticity Dependence of the Growth rate for cos- and sin-mode.